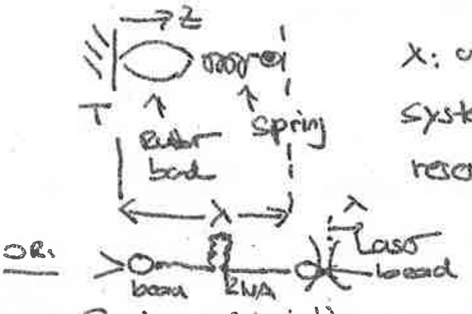
Jarzynski Equality

Basic notions to consider. (C. Jarzynski Example) "Macroscopic" thermodynamics



λ: work parameter

system: spring & band

reservoir: temperature T

(RNA Hairpin)

$\Delta U = W + Q$ ("First law")

heat generated

$W = \int F_{spring}\, dz = \int F_{spring}\, d\lambda$

Review: (brief)

2nd law of thermodynamics (S: Entropy) $\boxed{\text{Total entropy can never decrease over time (Carnot)}}$

$$\int_A^B \frac{dQ}{T} \le \Delta S \qquad \Delta S = S_B - S_A$$

T: Temperature of the RESERVOIR.

Equal sign: reversible processes (< corresponds to irreversible process)

Start in equilibrium $\lambda = A$ to $\lambda = B$ $(A \to B)$ and hold $\lambda = B$ fixed afterwards:

$$\boxed{W \ge \Delta F = F_B - F_A}$$

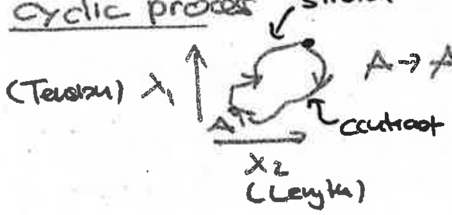
CLAUSIUS RELATION

2nd Law of thermodynamics (for 1 single reservoir) $T = \text{const}$ ISOTHERMAL PROCESS

$F = U - TS$ Helmholtz free energy.

(U: internal energy of system)

cyclic process



$A \to A$ $\quad W_{cycl} \ge 0$

$Q \le 0$ (heat is always generated)

consider forward & backward processes

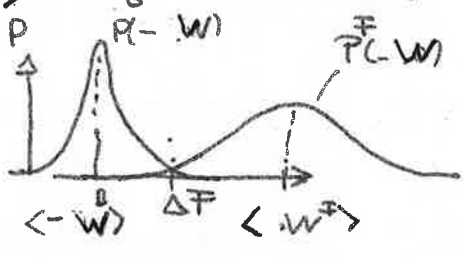
Forward $\lambda: A \to B$ (forward) $\quad W^F \ge F_B - F_A = \Delta F$

Backward $\lambda: B \to A$ (backward, reverse) $\quad W^B \ge F_A - F_B = -\Delta F$

combined this implies: $\underline{-\langle W^B \rangle \le \Delta F \le \langle W^F \rangle}$

graphically: ($\underline{\langle W^F \rangle + \langle W^B \rangle \ge 0}$ ie work done)

microscopic system.



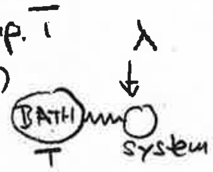
Probability of work.

Now: what happens for a microscopic system?

Jarzynski equality and Fluctuation theorems can be derived within the framework of the mesoscopic Master equation.

JARZYNSKI EQUALITY

Consider a system in contact with a thermal reservoir at temp. T



assume next the system is coupled to external parameter (λ)
e.g spin to magnetic field; gas to a piston .. etc.
Assume the parameter $\lambda \in [0..1]$ is switched from $\lambda = 0$ to $\lambda = 1$ slowly (adiabatically). The WORK performed is (for infinitely slow change)

- then $\boxed{W_{\infty} = \Delta F = F_1 - F_0}$ (Difference in the free energy)

or if it is done non adiabatically:

$$\langle W \rangle \geq \Delta F$$

$\langle . \rangle$ ensemble average $\langle W \rangle = \int dW \cdot p(W, t_s) W$

- The difference $\langle W \rangle - \Delta F = W_{diss}$ i.e the dissipated work associated with increase in entropy during irreversible process

The Jarzynski EQUALITY STATES that the FREE ENERGY DIFFERENCE ΔF for any switching time from $\lambda = 0$ to $\lambda = 1$ is:

$$\langle \exp[-\beta W] \rangle \hat{=} \int dW p(W, t_s) \cdot \exp(-\beta W) = \exp[-\beta \Delta F]$$

$$\beta \equiv (kT)^{-1}$$

i.e

$$\boxed{\langle \exp[-\beta W] \rangle = \exp(-\beta \Delta F)}$$

this equation is an EQUALITY and thus permits to find free Energy difference ΔF from stat. average of non-equilibrium process.

The equality can be proved with Master Equation (→ J. Jarzynski PRE 1997)

or:

$$\boxed{\Delta F = -\beta^{-1} \ln(\langle \exp(-\beta W) \rangle)}$$

One problem is that the exp-function makes the stat. average very sensitive to rare events (e.g large values of W)

→ Solution is CROOK'S EQUALITY

Non Equilibrium
Stat. Mech (2019)

Reference: Jarzinsky PRL 1996

<u>Review</u>: Fluctuation theorem of CRDOK and Jarzynski's equality

$$\boxed{\frac{P_u(W)}{P_R(-W)} = \exp\left(\frac{W-\Delta F}{k_B T}\right)} \qquad \text{CFT}$$

Here u, R denote well defined initial and final states (e.g. folded and unfolded in AFM molecular folding experiments) which are in equilibrium.

$P_u(W)$: probability distribution of the values of work performed during process u (e.g "unfolding")

$P_R(W)$: Prob. dist. of work performed during time reversed process (e.g "reverse" folding process)

ΔF free energy change between final and initial state (ΔF) (i.e Equilibrium freeenergy difference)

Thus: CFT predicts symmetry relation in the work fluctuations with forward and reverse changes, as system is driven out of equilibrium. <u>Note</u>: CFT and Jarzynski can be derived from each other (J. from CFT).

<u>Jarzynski's equality</u>

$$\boxed{\exp\left(-\frac{\Delta F}{k_B T}\right) = \left\langle \exp\left(-\frac{W}{k_B T}\right)\right\rangle}$$

$\langle .. \rangle$ average over repeated measurements / stat. average

where $\langle\rangle$ denotes a series of <u>non-equilibrium</u> measurements.

- For process that occur near equilibrium $\langle W \rangle = \Delta F$
  but: such experiments are sensible to <u>drift</u> (i.e change in apparatus/measurement protocol during time)
- The exp - function makes $\langle .. \rangle$ sensitive to rare events where W is large

Solution and importance of the CFT: one can perform force extension (i.e variation of external parameter) fast – non-equilibrium process, but at the expense of irreversible process.

CFT overcomes these problems!

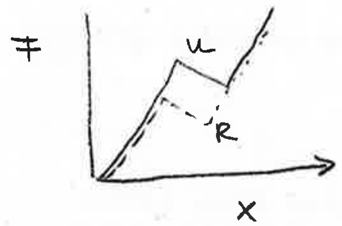
A consequence of the CFT is:

$$\boxed{\frac{P_u(W)}{P_R(-W)} = 1 \Rightarrow W = \Delta F}$$ free energy difference

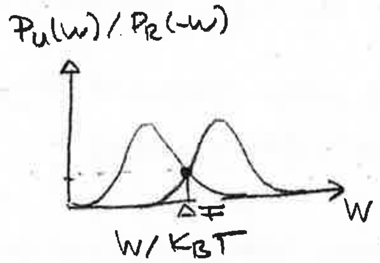
irrespective of the control parameter λ_n / speed of the protocol

Hence, if the distributions $P_u(W)$ and $P_R(-W)$ are known and recorded one can determine ΔF

Recording the entire work function increases the precision of the non-equilibrium free energy estimate.



stochastic Force distance curves.



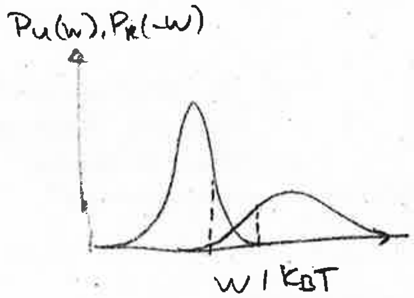
Work:

$W = \sum_n F_n \Delta x$

CROOKS FLUCT. THEOREM

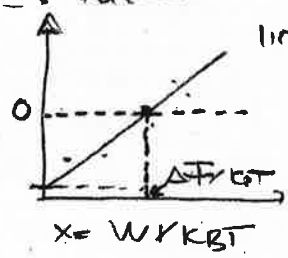
Problem: overlap decreases! Consider now eq(1):

$$\boxed{\frac{P_u(W)}{P_R(-W)} = \exp\left(\frac{W-\Delta F}{k_BT}\right)}$$



CFT →

$$\ln\left(\frac{P_u(W)}{P_R(-W)}\right) = \left(\frac{W-\Delta F}{k_BT}\right)$$



linear relationship

i.e. $\ln\left(\frac{P_u(W)}{P_R(-W)}\right) = x - \frac{\Delta F}{k_BT}$

→ intercept yields ΔF.

From: Bustamante Group (D. Collins, Nature, 2005)

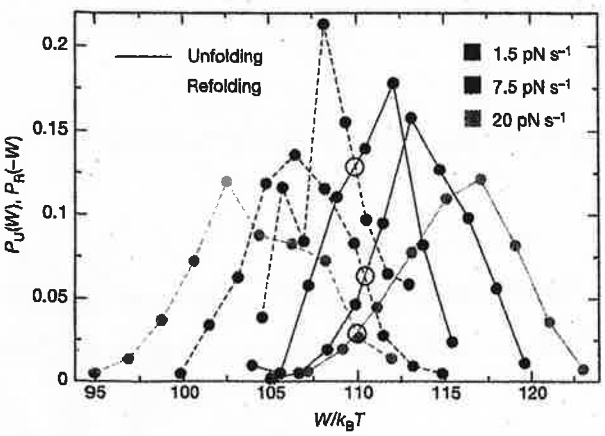
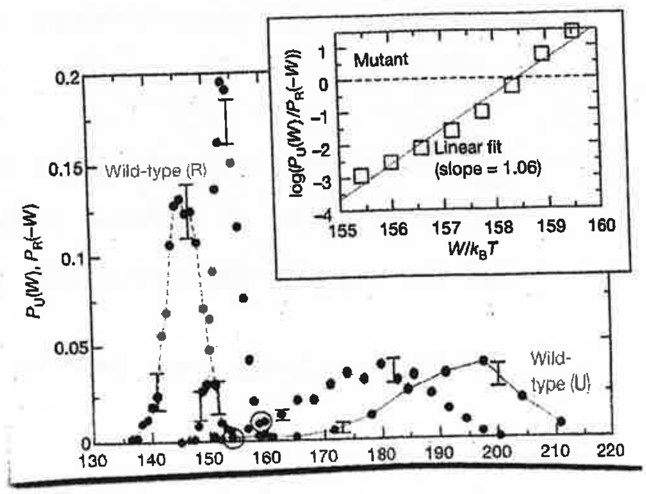


Figure 2 | Test of the CFT using an RNA hairpin. Work distributions for RNA unfolding (continuous lines) and refolding (dashed lines). We plot negative work, $P_R(-W)$, for refolding. Statistics: 130 pulls and three molecules ($r = 1.5\,\mathrm{pN\,s^{-1}}$), 380 pulls and four molecules ($r = 7.5\,\mathrm{pN\,s^{-1}}$), 700 pulls and three molecules ($r = 20.0\,\mathrm{pN\,s^{-1}}$), for a total of ten separate experiments. Good reproducibility was obtained among molecules (see Supplementary Fig. S2). Work values were binned into about ten equally spaced intervals. Unfolding and refolding distributions at different speeds show a common crossing around $\Delta G = 110.3\,k_BT$.



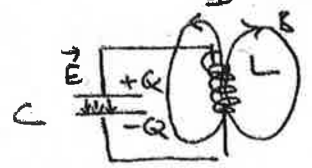
Review: quantization of an electrical circuit

Harmonic oscillator quantization: example

$\Phi = \int \vec{B}\, d\vec{A}$

Φ: Flux

Quantization of an electrical LC circuit.



Classical Energy: $\boxed{H = \underbrace{\frac{Q^2}{2C}}_{\text{electric energy}} + \underbrace{\frac{\Phi^2}{2L}}_{\text{magnetic energy}}}$

Φ: magnetic flux

Q: charge on capacitor $(Q = V \cdot C)$

$V = \dot{\Phi}$ Voltage

$I = \dot{Q}$ current

Equation of motion: $V = 0 = \underbrace{-L \cdot \dot{I}}_{V_{ind}} + \frac{Q}{C}$

Solution: $V(t) = \{\cos(\omega t), \sin(\omega t)\}$ $\quad \omega_{LC} = \sqrt{\frac{1}{LC}}$ (resonance frequency)

Quantization: $\hat{\Phi} = \sqrt{\frac{\hbar L}{2C}}\,(a + a^\dagger)$, $\quad [a, a^\dagger] = 1$

$\hat{Q} = \sqrt{\frac{\hbar C}{2L}}\,\left(\frac{a - a^\dagger}{i}\right)$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$

$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

thus

$\hat{a} \,\hat{=}\, \left(\sqrt{\frac{L}{C}}\,\hat{Q} + i\,\hat{\Phi}\sqrt{\frac{C}{L}}\right)\frac{1}{\sqrt{2\hbar}}$

(annihilation operator)

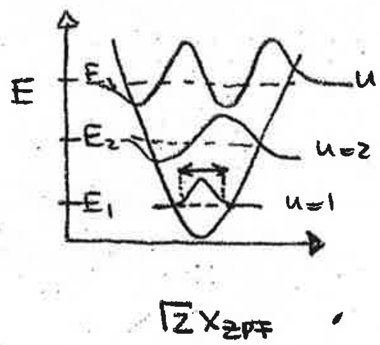
thus: $\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L} = \frac{\hbar}{(LC)^{1/2}}\left(\hat{Q}^2\sqrt{\frac{L}{C}} + \hat{\Phi}^2\sqrt{\frac{C}{L}}\right)\frac{1}{2\hbar}$

$\Leftrightarrow \quad \hat{H} = \left(\frac{1}{LC}\right)^{1/2}\hbar\, a^\dagger a + \frac{\hbar\omega_{LC}}{2} = \hbar\omega_{LC}\, a^\dagger a + \underbrace{\frac{\hbar\omega_{LC}}{2}}_{\text{zero point motion}}$

$[\hat{\Phi}, \hat{Q}] = i\hbar$

$Q_{ZPF} = \langle \hat{Q}^2 \rangle_{n=0} = \sqrt{\frac{\hbar L}{2C}}$ Vacuum charge fluctuation

(Electric Voltage fluctuation $V_{ZPF} = Q_{ZPF}/C$)



coherent states: $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle$

$(a|\alpha\rangle = \alpha|\alpha\rangle$ Eigenstate of "a")

Non Equilibrium Stat Mech

Dissipation in Quantum Mechanics

overview: derivation of a quantum Markov process leads
Quantum Langevin eqs and the Quantum Optical Master Equation.

REFERENCE
see: STOCHASTIC METHODS
C. GARDNER

Brief Review: Quantum Mechanics of Harmonic Oscillator

creation and annihilation (destruction operator)

$$\boxed{[\hat{a},\hat{a}^+] = \hat{a}\hat{a}^+ - \hat{a}^+\hat{a} = 1}$$ commutation relation

Eigenstates $\boxed{\hat{a}^+\hat{a}\,|n\rangle = n|n\rangle}$ $n \in \mathbb{N}_0$

(for proof see Cohen Tannoudji: Quantum Mechanics, Part I)

and:
$$\boxed{\begin{aligned} a|n\rangle &= \sqrt{n}\,|n-1\rangle \\ a^+|n+1\rangle &= \sqrt{n+1}\,|n+1\rangle \end{aligned}}$$

Orthogonality
$$\langle n|m\rangle = \delta_{m,n}$$

Number Operator
$$\boxed{\hat{N} = \hat{n} = \hat{a}^+\hat{a}}$$

Harmonic Oscillator: $\hat{H} = (a^+a + \frac{1}{2})\hbar\omega$

$\hbar$: Planck constant

$\frac{\hbar\omega}{2}$: Zero Point Energy

$$\boxed{\begin{aligned} &\hat{H}|n\rangle = E_n|n\rangle \text{ and Eigenenergy} \\ &E_n = (\hbar\omega)(n+\tfrac{1}{2}) \end{aligned}}$$

Schrödinger Equation

Physical state $|A,t\rangle$

$$H|A,t\rangle = i\hbar\partial_t|A,t\rangle \quad (\text{or } \hat{H}\psi(t) = -i\hbar\partial_t\psi(t))$$

One can expand state

$$|A,t\rangle = \underbrace{\sum_n |n\rangle\langle n|}_{1} A,t\rangle$$

$$\Leftrightarrow \quad i\hbar\partial_t|A,t\rangle = i\hbar\left(\sum_n |n\rangle\partial_t\langle n|A,t\rangle\right) = \sum_n H|n\rangle\langle n|A,t\rangle$$

$\Leftrightarrow$ $\langle n|A,t\rangle = e^{-iE_n t/\hbar} \langle n|A,0\rangle$ Evolution of system

e.g $|A,t=0\rangle = |1\rangle \to |A,t\rangle = |1\rangle e^{-i\omega t}$; $|A,t=0\rangle = |3\rangle = |3\rangle e^{-i3\omega t}$

COHERENT STATES (no proof)

Definition (Roy Glauber) $\boxed{|\alpha\rangle = \exp(-\frac{1}{2}|\alpha|^2) \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle}$

Scalar product $\langle\alpha|\beta\rangle = \exp(\alpha^*\beta - \frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2)$

$$|\langle\alpha|\beta\rangle|^2 = \exp(-|\alpha-\beta|^2)$$

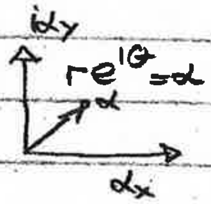
Thus coherent states of different eigenvalue are not orthorgonal They form an overcomplete basis

Completeness Formula:

$\boxed{\mathbb{1} = \frac{1}{\pi}\int_{\mathbb{C}} d^2\alpha\, |\alpha\rangle\langle\alpha|}$ $\mathbb{C}$: complex plane

Proof: assume general state $|A\rangle = \sum_n A_n|n\rangle$

Note $\alpha = \alpha_x + i\alpha_y$



$\Leftrightarrow$ $\frac{1}{\pi}\int d\alpha^2 (|\alpha\rangle\langle\alpha|)A\rangle = \int \frac{1}{\pi} \sum_n d^2\alpha A_n |\alpha\rangle\langle\alpha|n\rangle$

substitute polar coordinates $\alpha = re^{i\theta}$ $d^2\alpha = r\,dr\,d\theta$

Thus:

$$\frac{1}{\pi}\sum_{m,n}\int_0^\infty\int_0^{2\pi} A_n e^{-r^2} r^{n+m} e^{i(m-n)} (n!m!)^{-1/2} |m\rangle r\,dr\,d\theta$$

$$= 2\sum_n \int_0^\infty A_n e^{-r^2} r^{2n+1} (n!)^{-1} |n\rangle dr$$

where we used: $2\delta_{n,m} = \frac{1}{\pi}\int_0^{2\pi} e^{i(m-n)\theta} d\theta$

Thus: and using

$$\int_0^\infty dr\, e^{-r^2} (r^{2n+1}) = n!/2$$

$\Leftrightarrow$ $= 2\cdot(n!)^{-1}\left[\frac{n!}{2}\right] = 1$ Thus identity $\mathbb{1}$ q.e.d

Side Note $\boxed{\|\alpha\rangle = \exp(\frac{1}{2}|\alpha|^2)|\alpha\rangle}$ are called BARGMANN STATES

Expansion of an operator in coherent states ($\hat{T}$)

$$\hat{T} = \mathbb{1}\,\hat{T}\,\mathbb{1} = \frac{1}{\pi^2}\iint d^2\alpha\, d^2\beta\; |\alpha\rangle \underbrace{\langle\alpha|\hat{T}|\beta\rangle}_{*} \langle\beta|$$

$$= \frac{1}{\pi^2}\iint d^2\alpha\, d^2\beta\; |\alpha\rangle\langle\beta|\; T(\alpha^*,\beta)\exp\left(-\tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2\right)$$

where $* \quad T(\alpha^*,\beta) = \exp\left(\tfrac{1}{2}|\alpha|^2 + \tfrac{1}{2}|\beta|^2\right)\langle\alpha|T|\beta\rangle = \langle\alpha\|T\|\beta\rangle$

(BARGMANN STATES)

For example operator $\boxed{T = (\hat{a}^+)^m(\hat{a})^n}$

then, $T(\alpha^*,\beta) = \langle\alpha|(a^+)^m(\hat{a})^n|\beta\rangle \exp\left(\tfrac{1}{2}|\alpha|^2 + \tfrac{1}{2}|\beta|^2\right)$

$$= (\alpha^*)^m \beta^n \langle\alpha|\beta\rangle \exp\left(\tfrac{1}{2}|\alpha|^2 + \tfrac{1}{2}|\beta|^2\right)$$

(note that $\langle\alpha|a^+ = \alpha^*\langle\alpha|$)

$$\boxed{T(\alpha^*,\beta) = (\alpha^*)^m(\beta)^n \exp(\alpha^*\beta)}$$

Use this to calculate its Expectation value.

$$\langle\alpha|T|\alpha\rangle \underset{*}{=} \sum_{n,m}\langle n|T|m\rangle\, e^{-|\alpha|^2}(\alpha^*)^n(\alpha)^m\frac{1}{\sqrt{n!}\sqrt{m!}}$$

(replace $|\alpha\rangle$ by Fock state expansion)

so that $\langle n|T|m\rangle = \sqrt{n!}\sqrt{m!}\,\dfrac{\partial^n}{\partial\alpha^{*n}}\dfrac{\partial^m}{\partial\beta^m}\left(e^{\alpha\alpha^*}\langle\alpha|T|\alpha\rangle\right)$

Formal derivatives in ANALYTIC FUNCTION THEORY

$$\begin{cases} \alpha = x + iy \\ \dfrac{\partial}{\partial\alpha} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right) \\ \dfrac{\partial}{\partial\alpha^*} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right) \end{cases}$$

COHERENT STATES

$$\boxed{a|\alpha\rangle = \alpha|\alpha\rangle \quad \text{and} \quad \langle\alpha|a^+ = \alpha^*\langle\alpha|}$$

NORMALLY ORDERED OPERATORS

$\langle\alpha|a^+aa^+|\beta\rangle$ is not normally ordered

$\langle\alpha|a^+aa^+|\beta\rangle = \langle\alpha|\,a^+(a^+a+1)\,|\beta\rangle \qquad [a,a^+]=1$

$= \alpha\langle\alpha|\underbrace{a^+a^+a}_{\text{Normally ordered}} + a^+|\beta\rangle \qquad aa^+ - a^+a = 1$

$= (\alpha^{*2}\beta + \alpha^*)\langle\alpha|\beta\rangle$

LECTURE 10

": :" Normal ordering (important in e.g. optical photodetection)

$$:(a+a^\dagger)(a+a^\dagger): = a^{\dagger 2} + a^2 + 2a^\dagger a$$

(i.e. destruction operators to right of creation operators)

Poisson number distribution of coherent state

$|n\rangle$: number state

$$\boxed{P_\alpha(n) = |\langle n|\alpha\rangle|^2 = \left|\exp(-\tfrac{1}{2}|\alpha|^2)\frac{\alpha^n}{\sqrt{n!}}\right|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}}$$

Poisson distribution with a mean $|\alpha|^2$

$$\langle \hat N\rangle = \langle\alpha|\hat N|\alpha\rangle = \sum_n nP(n) = \dots = |\alpha|^2$$

$$\langle \hat N^2\rangle = \langle\alpha|a^\dagger a a^\dagger a|\alpha\rangle$$

$$= \langle\alpha|a^\dagger(a^\dagger a+1)a|\alpha\rangle = \langle\alpha|a^\dagger a^\dagger aa + a^\dagger a|\alpha\rangle$$

$$= |\alpha|^4 + |\alpha|^2$$

Thus: $\langle \hat N(\hat N-1)\rangle = |\alpha|^4 = \langle \hat N\rangle^2$ i.e. $\boxed{\langle \hat N^2\rangle - \langle \hat N\rangle = \langle \hat N\rangle^2}$

PROPERTY OF A POISSON PROCESS

REVIEW

DENSITY MATRIX AND PROBABILITIES

$\langle \hat M\rangle = \langle\psi|\hat M|\psi\rangle$ expectation value of operator $\hat M$

For a non pure state

$$\langle \hat M\rangle = \sum_a P(a)\langle\psi_a|\hat M|\psi_a\rangle$$

Definition of the density Matrix $\boxed{\rho \equiv \sum_a P(a)|\psi_a\rangle\langle\psi_a|}$

Then: $\boxed{\langle M\rangle = \mathrm{Tr}\{\rho M\} \quad \text{where } \mathrm{Tr}\{B\} \equiv \sum_n \langle n|B|n\rangle}$

Thus since:

$$\mathrm{Tr}\{\rho M\} = \sum_{n,a} P(a)\langle n|\psi_a\rangle\langle\psi_a|\hat M|n\rangle$$

$$= \sum_{n,a} P(a)\langle\psi_a|M|n\rangle\underbrace{\langle n|\psi_a\rangle}_{\hat{=}\,|\psi_a\rangle}$$

since: $|\psi_a\rangle = \sum_n \langle n|\psi_a\rangle|n\rangle$

$$= \sum_a P(a)\langle\psi_a|M|\psi_a\rangle$$

$$= \langle M\rangle$$

LECTURE 10

$Tr\{\rho M\} = Tr\{M\rho\}$ cyclic property of the trace

Properties of the density Matrix

i) $Tr\{\rho\} = 1 = \sum_a P_a \langle\psi_a|\psi_a\rangle = \sum P_a = 1$ (sum of diagonal elements "Populations")

(ii) ρ is semi definative for state $|A\rangle$

$\langle A|\rho|A\rangle = \sum P_a |\langle A|\psi_a\rangle|^2 \geq 0$

(iii) For a PURE state $\rho^2 = \rho$ and thus conversely $P_a = \delta_{a,a_0}$

$\left(\rho^2 = \sum_i P_a P_b |\psi_a\rangle\langle\psi_b|\langle\psi_a|\langle\psi_b|\right)$

(iv) $Tr\{\rho^2\} \leq Tr\{\rho\}$

Equation of Motion for ρ: VON NEUMANN'S EQUATION

$$\boxed{H|\psi\rangle = -i\hbar\partial_t|\psi\rangle} \qquad \rho = \sum_a p_a |\psi_a\rangle\langle\psi_a|$$

$$\partial_t\rho = \sum_i p_a [\underbrace{\partial_t|\psi_a\rangle}_{i\hbar\partial_t|\psi_a\rangle = H|\psi_a\rangle}\langle\psi_a| + |\psi_a\rangle\underbrace{\langle\partial_t\psi|}_{\Rightarrow -i\hbar\partial_t\langle\psi| = +\langle\psi|H}]$$

$$\partial_t\rho = \frac{\sum P_a}{i\hbar}(\hat{H}|\psi_a\rangle\langle\psi_a| - |\psi_a\rangle\langle\psi_a|\hat{H})$$

$$\boxed{i\hbar\,\partial_t\rho = (H\rho - \rho H) = [H,\rho]}$$ NEUMANN EQUATION ("QUANTUM LOUVILLE THEOREM")

$$\boxed{\rho(t) = \exp\left(-\frac{i\hat{H}t}{\hbar}\right)\rho(0)\exp\left(\frac{i\hat{H}t}{\hbar}\right)}$$

Glauber P representation GLAUBER-SUDARSHAN PHASE

assume that $\boxed{\rho = \int d^2\alpha\, \underbrace{P(\alpha,\alpha^*)}_{\text{"quasi probability density"}}|\alpha\rangle\langle\alpha|}$

Note that: $\int d^2\alpha\, P(\alpha\alpha^*) = 1$

since $1 = \langle\alpha|\alpha\rangle = Tr\{\rho\} = \int d^2\alpha\, P(\alpha,\alpha^*)\underbrace{\sum_n \langle n|\alpha\rangle\langle\alpha|n\rangle}_{=1}$

OPERATOR CORRESPONDENCE

$a|\alpha\rangle = \alpha|\alpha\rangle$ $\qquad a^{+}||\alpha\rangle = \sum_n \frac{\alpha^n}{\sqrt{n!}}\sqrt{n+1}\,|n+1\rangle = \frac{\partial}{\partial\alpha}||\alpha\rangle$

and: $\boxed{\langle\alpha||a = \frac{\partial}{\partial\alpha^*}\langle\alpha||}$ $\qquad \boxed{\left(\langle\alpha|a^* = \langle\alpha|\alpha^*\right)}$

operator correspondences

$$a^{+}\rho = a^{+}\int d^2\alpha\, ||\alpha\rangle\langle\alpha||\, e^{-\alpha\alpha^*} P(\alpha,\alpha^*)$$

$$= \int \partial^2\alpha\, \frac{\partial}{\partial\alpha}||\alpha\rangle\langle\alpha||\, e^{-\alpha\alpha^*} P(\alpha,\alpha^*)$$

$$= \int d^2\alpha\, ||\alpha\rangle\langle\alpha||\, e^{-\alpha\alpha^*}\left(\alpha^* - \frac{\partial}{\partial\alpha}\right)P(\alpha,\alpha^*) \qquad \text{I. BY parts.}$$

$a\rho \to \alpha P(\alpha,\alpha^*)$	OPERATOR CORRESPONDENCE
$a^{+}\rho \to \left(\alpha^* - \frac{\partial}{\partial\alpha}\right) P(\alpha,\alpha^*)$	(This will be needed
$\rho a \to \left(\alpha - \frac{\partial}{\partial\alpha^*}\right) P(\alpha,\alpha^*)$	later to solve the
$\rho a^{+} \to (\alpha^*) P(\alpha,\alpha^*)$	Quantum Master Equation)

Application: Driven Harmonic Oscillator

$$\hat{H}_0 = \hbar\omega\left(a^{+}a + \tfrac{1}{2}\right)$$

$$\hat{H}_{DRIVE} = \left(\lambda a^{+} + \lambda^* a\right)$$

Hence:

$$i\hbar\partial_t\rho = \hbar\omega[a^{+}a,\rho] + \lambda[a^{+},\rho] + \lambda^*[a,\rho]$$

Can be converted to a differential equation:

$$\partial_t P(\alpha,\alpha^*) = -i\left(\omega\frac{\partial}{\partial\alpha}\cdot\alpha + \omega\frac{\partial}{\partial\alpha^*}\alpha^* - \frac{\lambda}{\hbar}\frac{\partial}{\partial\alpha} + \frac{\lambda^*}{\hbar}\frac{\partial}{\partial\alpha^*}\right)P$$

solution:

$$P(\alpha,\alpha^*,t) = \partial^2(\alpha - \alpha(t))$$

$$\alpha(t) = \alpha(0)e^{i\omega t} + i\int_0^t dt'\, e^{-i(\omega(t-t'))}\lambda(t')/\hbar$$